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MATHEMATICAL MODELLING OF TONE-DENSITY REPRODUCTION IN FLEXOGRAPHIC PRINTING ON METALLIZED FOIL

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Abstract

Metallized foil is a strategically important substrate in the flexible packaging for food and pharmaceuticals due to its superior barrier properties and mirror-like appearance. However, its non-absorbent, highly specular surface creates significant printing challenges, as its optical behavior differs fundamentally from conventional paper. Despite growing demand for color consistency in premium packaging, systematic studies of the full CMYK tone-density response on metallized foil remain scarce. Existing color management protocols, developed for paper, require significant adaptation for specular surfaces. This study aims to provide a comprehensive experimental and mathematical characterization of CMYK tone-density reproduction on metallized BOPP foil under controlled industrial conditions.

Low-migration CMYK inks were flexo-printed on metallized BOPP, with 1–100 % tone scales for each channel measured via spectrodensitometer per ISO 5-3 and ISO 12647-6. Three complementary mathematical models were applied to analyze the experimental data: the Bouguer–Beer–Lambert law, extended to account for the dual optical path length on specular reflective substrates; the substrate-corrected Murray–Davies equation for optical dot-area estimation and tone value increase (TVI) calculation; and the Yule–Nielsen modified equation with an empirical n-factor fitted independently for each CMYK channel. Additionally, a Boltzmann sigmoidal regression model was fitted to each complete tone-density curve.

The D_{max} hierarchy C > M > Y > K resulted from the dual optical path length enhancement on the specular aluminum background, as defined by the Bouguer–Beer–Lambert model. Calculations using the substrate-corrected Murray–Davies equation revealed a significant tone value increase (TVI) of 29–36% in the midtones, which substantially exceeds typical values for paper due to the high reflectance contrast of the foil. The fitted Yule–Nielsen factors (n=1.18–1.35) characterized the effective optical response of the printed ink–foil system, while the Boltzmann sigmoidal regression model described the tone-density curves with high precision (R^2 = 0.9968–0.9981 across the four CMYK channels).

The inverted form of this model provides a direct mathematical tool for constructing tone compensation curves applicable in RIP-based tone-linearization workflows. The results demonstrate that tone reproduction on metallized foil is governed by substrate-specific optical mechanisms that require dedicated mathematical treatment, and cannot be adequately predicted by models calibrated on paper-based printing systems.
Keywords: *flexographic printing, metallized foil, optical density, Murray–Davies equation, Yule–Nielsen model, flexible packaging.*

Introduction

Metallized biaxially oriented polypropylene (mBOPP) and metallized polyethene terephthalate (mPET) foils are extensively used in flexible packaging for food, confectionery, pharmaceutical and cosmetic products. Their mirror-like aluminum metallization layer provides an outstanding barrier against moisture, oxygen, and UV radiation while creating a premium visual effect. However, the same specular surface that imparts these functional and aesthetic advantages fundamentally alters the optical behavior of printed ink films, making standard densitometric and colorimetric models developed for paper substrates inadequate for quality prediction and process control [1-3].

Recent flexographic research has shown that the opacity of an underlying white ink layer affects the color of inks printed above it [4]. Studies of multilayer aluminum-based reflective structures have also demonstrated that their optical appearance depends on the balance between specular and diffuse reflection [5]. These findings underline the need to specify the layer structure when interpreting measurements on metallized materials.

In conventional printing on white paper, optical density measurements are governed primarily by ink-film absorption and surface scattering within the paper fibers. On metallized foil, the specular reflectance of the aluminum layer introduces an additional optical path component, altering the effective absorption geometry and rendering the classic Bouguer–Beer–Lambert model insufficient without modification. Furthermore, because the foil surface is non-absorbent, ink spreading behavior differs significantly from that on paper, affecting dot gain and tone reproduction in ways not captured by empirical models calibrated on cellulosic substrates.

The objective of the present study is to characterize tone-density reproduction for CMYK low-migration flexographic inks on metallized BOPP foil under controlled industrial printing conditions, and to develop and validate mathematical models that accurately describe the observed tone-density relationships.

Methodology and equipment

The substrate was a three-layer metallized foil formed by vacuum deposition of a nanoscale aluminum layer onto a biaxially oriented polypropylene (BOPP) polymer base – a widely used construction in flexible packaging metallization. Printing was performed on an Omet Varyflex V2 670 flexographic press equipped with a hot-air drying system (HAD convection system), using low-migration CMYK process inks manufactured by Huber Group. During flexographic printing on metallized foil, an initial white ink layer is applied to create an opaque background and ensure accurate color reproduction. However, white ink is frequently applied selectively rather than over the entire area. Where the white layer is absent, process inks are laid down directly onto the foil, creating an attractive "metallic" effect. Therefore, research and testing of layout areas where the opaque white layer is intentionally omitted to achieve the "metallic" effect represent a critical stage of the technological process. The necessity for such research stems from the specific interaction between printing inks and the mirror-like, non-porous surface of the foil.

The CMYK tone scales investigated in this work were printed directly onto the metallized BOPP film without a white background. All densitometric measurements and mathematical analyses presented below refer exclusively to these tone scales.

Printing plates were produced on a Kodak Flexcel NX imager (CTF technology) with UV exposure on a Vianord Engineering unit. The CMYK image plates used Kodak DigiCap A00 microcell screening (Standard) combined with Kodak HyperFlex NX technology for dot support up to 55% screen value, while the white base coat plate used Kodak DigiCap A02 microcell screening. Anilox rolls with lineature 700 lpi and cell volume 2.8 cm^3/m^2 (laser-engraved ceramic, 60° cell angle) were used for all printing units.

A tone scale from 1% to 100% (AM screening, 150 lpi) was printed for each CMYK channel. Reflection density was measured with an X-Rite eXact spectrodensitometer [6] using 45°:0° measurement geometry and Status T density response per ISO 5-3:2009 [7, 8]. For each CMYK channel, the reference density D_0 was estimated from the measured density of the corresponding 1% tone patch, the lowest tone value included in the printed scale. For each tone patch, one measurement was taken from each of five produced imprints, and the arithmetic mean of the five measurements was used for the analysis. All computations were performed in Python 3.11 using NumPy and SciPy libraries for curve fitting.

Three theoretical frameworks are applied and compared: the Bouguer–Beer–Lambert law, the Murray–Davies equation, and the Yule–Nielsen modification. Additionally, a sigmoidal (Boltzmann) regression model is

fitted to the experimental data to provide a practical predictive tool for prepress engineers and color management specialists [9–11].

For a solid (100%) ink patch, the optical density D of a transparent ink film of thickness d on a reflective substrate is described by the Bouguer–Beer–Lambert law:

$$D = \varepsilon \cdot c \cdot d \quad (1)$$

where ε is the molar attenuation coefficient of the ink ($L \cdot mol^{-1} \cdot cm^{-1}$), c is the ink concentration ($mol \cdot L^{-1}$), and d is the optical path length through the ink film (cm). On a specular metallic substrate measured via 45°:0° geometry, the directional specular component is largely excluded from the direct receiver angle, meaning that light interaction involves scattering and internal reflections within the ink film layer. Consequently, the measured reflection density D_r of the printed ink layer is described by an empirical modification of the Bouguer–Beer–Lambert law:

$$D_r = 2 \cdot \varepsilon \cdot c \cdot d + D_0 \quad (2)$$

where D_0 represents the effective baseline optical density evaluated from the lowest tone patch (1% step) to account for the local microstructural response of the unprinted metallized substrate. The factor of two represents an effective optical path scaling parameter in this empirical model. The experimentally determined D_{max} values (Y = 1.43, M = 1.54, C = 1.60, K = 1.41) are consistent with this model, with cyan reaching the highest value due to its broader spectral absorption band and favorable interaction with the aluminum reflectance spectrum.

The apparent (optical) dot area a(t) at a given tone step is calculated using a substrate-corrected modification of the classical Murray–Davies equation [9]:

$$a(t) = \frac{10^{-D_t} - 10^{-D_0}}{10^{-D_s} - 10^{-D_0}} = \frac{R_t - R_0}{R_s - R_0} \quad (3)$$

where D_t is the measured optical density at nominal tone value t, $D_s = D_{max}$ is the solid ink density, and D_0 is the residual density of the unprinted metallized foil substrate. Expressed in terms of reflectance ($R_0 = 10^{-D0}$, $R_s = 10^{-Ds}$, and $R_t = 10^{-Dt}$, this equation ensures a = 0 for the unprinted substrate and a = 1 for the solid ink layer.

Tone value increase (TVI) in percent is then defined as:

$$TVI(t) = \left(a(t) - \frac{t}{100}\right) \cdot 100\% \quad (4)$$

Table 1 presents TVI values calculated at representative tone steps for all four CMYK channels. It must be noted that the classical Murray–Davies equation (Eq. 3) was originally developed for white paper substrates, where the substrate reflectance is assumed to be unity ($D_o = 0$). On metallized BOPP foil, the unprinted substrate exhibits a non-negligible residual density D_o, estimated from the 1% tone patches as: $D_o(Y) = 0.160$, $D_o(M) = 0.165$, $D_o(C) = 0.160$, $D_o(K) = 0.116$.

Table 1. Substrate-corrected TVI (%) for CMYK inks on metallized BOPP foil (Eq. 3)

Tone, %	TVI Y, %	TVI M, %	TVI C, %	TVI K, %
25	26.4	28.2	27.5	29.1
30	31.2	32.8	31.4	33.5
35	34.8	36.2	35.1	35.9
40	33.1	34.0	33.2	34.2
45	28.5	29.4	28.8	29.8
50	21.0	22.1	21.5	22.4
60	12.4	13.2	12.8	13.5
75	4.5	5.0	4.7	5.2

The Murray–Davies equation does not account for the scattering of light within the ink film and substrate, which causes light to emerge from beneath dot edges and artificially increases the apparent dot area. Yule and Nielsen proposed a correction introducing an empirical factor n:

$$a_{YN}(t) = \frac{R_t^{1/n} - R_0^{1/n}}{R_s^{1/n} - R_0^{1/n}}, R_0 = 10^{-D_0}, R_t = 10^{-D_t}, R_s = 10^{-D_s} \quad (5)$$

where R_0, R_t, and R_s are the reflectances of the reference substrate, tone patch, and solid ink patch, respectively, as defined for Eq. (3), and $n \geq 1$ is the Yule–Nielsen factor. At n=1, Eq. (5) reduces to the substrate-corrected Murray–Davies equation (3). The Yule–Nielsen factor was originally introduced to account for optical effects in halftone printing on paper. In the present study, n is used as an effective correction factor for the optical response of the printed ink–foil system. Accordingly, its fitted values should be interpreted for the specified ink, substrate, and measurement conditions.

The factor n was determined empirically for each CMYK channel by minimizing the root-mean-square error (RMSE) between the dot area calcu-

lated using the Yule–Nielsen equation and the nominal tone value t/100. The fitted n is treated as an effective parameter of the printed ink–foil system. The difference between the calculated apparent dot area and the nominal tone value represents total tone value increase, encompassing optical and mechanical contributions. The best-fit n-values obtained were:

$$n^Y = 1.22, \quad n^M = 1.28, \quad n^C = 1.18, \quad n^K = 1.35 \quad (6)$$

The higher n-factor for black (K) reflects its broader spectral absorption and stronger interaction with the metallic substrate, while cyan (C) exhibits the lowest n owing to its narrower absorption band and more spectrally selective interaction with the aluminum reflectance. All values fall within the range 1.15–1.35, consistent with the physical expectation for scattering confined to the ink film on a specular non-absorbent substrate, and markedly lower than typical paper values ($n_{paper} \approx 1.6–2.0$). Table 2 compares the apparent dot area a^{YN} calculated using the substrate-corrected Murray–Davies equation (n = 1.0), the empirically optimized Yule–Nielsen model ($n = n_{opt}$), and a comparative high-scattering scenario (n = 2.0) across selected tone values [9].

Table 2. Apparent dot area a^{YN} (%) calculated with the substrate-corrected Yule–Nielsen equation at selected tone values and n-factors

n value	Tone, %	a	a	a	a
n = 1	25	51.4	53.2	52.5	54.1
n = 1	50	71.0	69.7	70.2	72.4
n = 1	75	88.4	87.6	88.0	89.2
n (opt.)	25	34.5	38.8	36.9	44.1
n (opt.)	50	67.0	65.4	64.5	69.9
n (opt.)	75	84.9	84.3	84.3	85.9
n = 2	25	28.1	29.3	30.1	32.6
n = 2	50	61.0	58.5	59.7	62.1
n = 2	75	81.9	80.6	81.9	82.1

The complete tone-density curves for all four channels exhibit a characteristic sigmoidal shape. A Boltzmann sigmoidal function was used to describe optical density D as a function of nominal tone value t over the measured range of 1–100%:

$$D(t) = D_{max} / [1 + exp(-(t - t_o) / \tau)] \quad (7)$$

where D(t) is the predicted optical density at tone value t, D_{max} is the upper-asymptote parameter set equal to the measured density of the solid ink patch, t_o is the inflection point of the curve (tone value at which the rate of density increase is maximum), and τ is the slope parameter controlling the steepness of the transition. The Boltzmann function was selected as an empirical model because its sigmoidal shape describes the slower density change in the highlight and shadow regions and the steeper response in the midtones. Equation (7) does not impose $D(0)=D_0$ or $D(100)=D_{max}$; therefore, the model and its inverse are used for intermediate target densities within the characterized tonal range.

The fitted parameters obtained by non-linear least-squares optimization (Levenberg–Marquardt algorithm) are given in Table 3.

Table 3. Fitted Boltzmann sigmoidal parameters for each CMYK channel

Parameter	Yellow (Y)	Magenta (M)	Cyan (C)	Black (K)
D_{max}	1.43	1.54	1.60	1.41
t_o (%)	52.4	49.8	51.3	50.9
τ	18.7	17.2	19.4	16.8
R^2	0.9981	0.9974	0.9979	0.9968

All four CMYK channels are described by the Boltzmann model with R^2=0.9968–0.9981, confirming that the sigmoidal function is an excellent predictor of tone-density behavior on metallized foil. The inflection point $t_o \approx$ 50–52% corresponds physically to the tone step at which the rate of dot coverage increase per unit density change is greatest, consistent with the midtone sensitivity of halftone screens.

The practical value of Eq. (7) lies in its invertibility: given a target optical density D_t, the required nominal tone value t can be computed directly as:

$$t = t_o - \tau \cdot ln(D_{max} / D_t - 1) \quad (8)$$

Equation (8) provides the theoretical basis for constructing tone compensation (transfer) curves that linearize the tone-density response of the printing system for a given channel and substrate combination, and can be incorporated as a one-dimensional tone (transfer) curve in RIP software.

Presentation of research results

Figure 1 presents the tone-density reproduction curves for all four CMYK process inks printed on silver metallized BOPP foil, with optical density ex-

pressed as a fraction of the respective channel maximum (D/D~max~) plotted against nominal tone value from 1% to 100%. Discrete markers represent experimentally measured values; solid lines are fitted Boltzmann sigmoidal curves (Eq. 7); the dashed diagonal represents the ideal linear response.

All four channels exhibit a pronounced sigmoidal departure from linearity. In the highlight region (1–20% tone), the rate of density build-up is below the linear reference, indicating that small dot areas contribute disproportionately less optical density than geometrically expected – a consequence of the specular substrate's high base reflectance. In the midtone region (30–70%), the curves rise steeply and converge closely, reflecting the dominant contribution of dot coverage to optical density regardless of ink color. In the shadow region (85–100%), all curves plateau sharply, with the final 5% of nominal tone producing less than 2–3% additional relative density, consistent with ink-film saturation described by the Bouguer–Beer–Lambert model (Eq. 2). The close agreement between measured points and fitted Boltzmann curves (R^2=0.9968–0.9981 across the four CMYK channels) confirms that Eq. (7) is an accurate predictive model for tone-density behavior on metallized foil across the full tonal range.

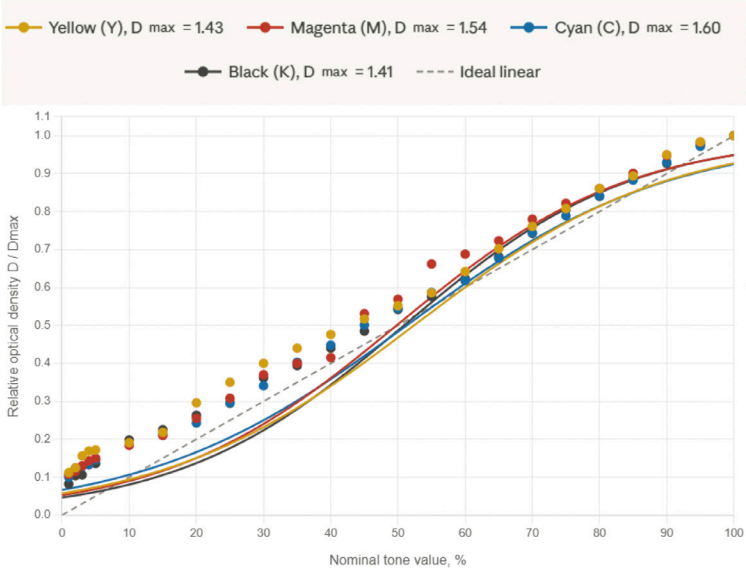


Fig. 1. Tone-density reproduction curves for CMYK inks on metallized BOPP foil. Markers: measured values. Solid lines: fitted Boltzmann curves. Dashed diagonal: ideal linear response

Cyan achieves the highest Dmax (1.60), followed by magenta (1.54), yellow (1.43), and black (1.41). This sequence – C > M > Y > K – differs from the typical hierarchy on white paper (K > C > M > Y) and reflects the combined effect of the dual-pass optical enhancement on the reflective substrate and the spectral absorption characteristics of each ink. In particular, cyan ink exhibits strong absorption in the red spectral region where the aluminum metallization has high reflectance, thereby maximizing the effective optical path enhancement defined in Eq. (2).

TVI values (Table 1) reveal that all four CMYK channels on metallized BOPP foil exhibit substantially elevated TVI values, peaking at 34.8–36.2% in the 35 nominal tone range. These values are computed using the substrate-corrected Murray–Davies form (Eq. 3), which accounts for the non-zero reflectance of the metallized foil substrate (D_o = 0.116–0.165 depending on the ink channel). In contrast to printing on white paper – where peak TVI is typically 10–18% at 50% nominal tone — the metallized substrate produces anomalously elevated optical dot gain. This is a direct physical consequence of the high reflectance contrast between the unprinted foil ($R_o \approx 0.68–0.77$) and the solid ink patches ($R_s \approx 0.025–0.039$). The resulting contrast ratio R_o/R_s is significantly higher than on white paper, which amplifies the optical contribution of each printed dot relative to its geometric area – a mechanism fundamentally different from the subsurface scattering that drives TVI on paper. In effect, the mirror-like inter-dot regions create an optical background that makes each dot appear optically larger than its physical dimensions, producing the elevated TVI observed across all channels. This finding has critical practical implications: standard tone compensation curves designed for conventional flexographic printing on film or paper substrates are entirely inadequate for metallized foil. Prepress workflows must incorporate substrate-specific TVI correction curves, directly derivable from the inverted Boltzmann model (Eq. 8), to achieve accurate tone reproduction on metallized packaging materials.

Conclusions

This study presents a comprehensive mathematical characterization of tone-density reproduction for CMYK low-migration flexographic inks on silver metallized BOPP foil. The principal conclusions are:

1. Three mathematical models – Bouguer–Beer–Lambert (Eq. 1–2), Murray–Davies (Eq. 3–4), and Yule–Nielsen (Eq. 5–6) – were applied and validated against experimental data. The Yule–Nielsen model with empirically optimized n-factors (Y: 1.22; M: 1.28; C: 1.18; K: 1.35), are effective parameters describing the optical response of the investigated ink–foil

system. The model accounts for this response when estimating apparent dot area under the specified printing and measurement conditions.

2. The Boltzmann sigmoidal regression model (Eq. 7, with R^2 ranging from 0.9968 for black to 0.9981 for yellow) accurately predicts the tone-density curve within the working tonal range (1–100%) and, in its inverted form (Eq. 8), provides a basis for constructing channel-specific tone compensation curves for RIP-based workflows.

3. The unusual Dmax hierarchy C > M > Y > K – inverse to that observed on paper – is attributed to the spectral selectivity of the Bouguer–Beer–Lambert dual-pass enhancement on a specular aluminum background.

4. Substrate-corrected TVI (Eq. 3) reaches peak values of 34.8–36.2% in the 35% nominal tone range for all CMYK channels – significantly higher than the 10–18% typically observed on white paper substrates. This elevated optical dot gain is driven by the high reflectance contrast between the unprinted metallized foil and the solid ink patches, and must be compensated in prepress via substrate-specific tone correction curves derived from Eq. (8).

The mathematical framework developed here supports tone curve optimization and provides densitometric data that may be used alongside additional colorimetric measurements in future ICC profile development for flexographic printing on metallized foil.

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